IV. "On the Motion under Gravity of Fluid Bubbles through Vertical Columns of Liquid of a different Density." By F. T. Trouton. Communicated by Professor Fitzgerald, F.R.S. Received May 3, 1893.

The kind of motion herein referred to can be observed by means simply of a glass tube, closed at one end, and provided with a stopper. If the tube be filled with water to nearly the top, closed, and then placed upside down, the enclosed bubble of air while ascending to the top can be observed, and the speed of ascent ascertained between two measured marks.

By enclosing different volumes of air it was found that the speed depended on the length of the bubble. The relation connecting the volume of the bubble with its speed of ascent was experimentally investigated. The speed, as will be seen from experiments subsequently described, may be taken within limits as a periodic function of the volume of the bubble. Bubbles greater than a certain thing all have the same velocity.

Experiments have also been made with other liquids. By mixing two liquids, such as water and glycerine, a series of determinations of speed with liquids of gradually increasing viscosity can be made. In these experiments the size of the bubble was outside the periodic limit. Contrary to expectation it was found that as the viscosity of water was increased by adding glycerine, the velocity increased instead of diminished. With tubes of about 0·7 cm. in diameter, it is not until the viscosity of the solution used is about eight times that of water that the velocity comes to be less than that through pure water. From this state the *velocity* was found to be *inversely proportional to the viscosity* of the solution, other things the same.

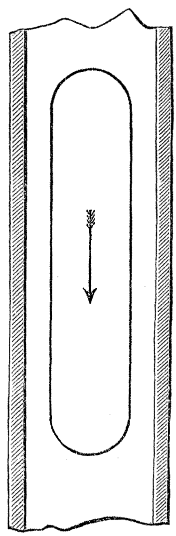
Instead of air the bubble may be of some liquid which does not too readily mix with that of which the column consists. In this way experiments were made to investigate the relation between the speed of ascent and the difference in density of the liquids, and also whether surface tension may have any influence.

The consideration of the subject may then conveniently be divided into two parts. The first part will deal with the dependence of the speed with which the bubble travels through the liquid column on the physical properties of the two fluids concerned in the phenomenon. The second part will refer to the connexion between the size or volume of the bubble and its speed.

PART I.

The physical properties involved in the phenomenon may, perhaps, be best studied by taking a particular case, say that of a bubble of chloroform falling through a column of glycerine contained in a glass tube. The appearance* is very much as shown in section in the figure.

FIG. 1.



In order that the bubble may descend, the liquid in front has to pass up the sides through the narrow annular space between the bubble and the tube. Were the diameter of the bubble known, the question would reduce itself to a case of viscous flow through an annular space—provided we neglected the ends and supposed the walls of the bubble to be rigid. The annular width will be seen to depend on the surface tension between the liquids, for should the tension become very great, say, the bubble must swell out, blocking up the tube. This tendency is in part counterbalanced by the excess in density of the bubble over that of the liquid column. The case is, so to speak, then that of viscous flow through an adjustable annular orifice.

The pressure per centimetre, or the pressure gradient driving up the liquid through this annular space, depends simply on the difference in density of the liquids.

* The length of the bubble was always several times its diameter, so as to get outside the stage where the velocity depends on the length of bubble.

The surface of the bubble moves by no means in a rigid manner, as can be seen by watching the movements of little particles of dust which may be present. The liquid of the bubble is seen in constant circulation flowing up the side with the current of glycerine, and returning down the centre of the bubble. Thus the viscosity of the liquid of which the bubble is composed must affect the velocity of its descent, but in what follows it has been neglected; this could be done, because the viscosity of the liquid of the column in most of the experiments was very great compared with that of the bubble.

Collecting the various things on which the velocity may depend, we have: 1, the pressure gradient, that is to say, the difference in density of the two liquids multiplied by the acceleration of gravity δg; 2, the viscosity of the liquids μ; 3, the surface tension between the liquids S; 4, the diameter of the tube D. It is difficult to see that anything else could come in to affect the rate of flow unless it be a slipping over the solid surface.

Thus we may put

$$\mathrm{V} = f(\delta g \mu \mathrm{SD}).$$

Assuming the function to have the form

$$\mathrm{V}^{-1} = \Sigma\mathrm{A}\,(\delta^x g^y \mu^z \mathrm{S}^m \mathrm{D}^n),*$$

we can obtain three equations from the considerations of dimensions to help determine the unknown exponents.

$$\begin{array}{ll} \text{From length,}\dagger & -1 = -3x+y-z+n, \\ \text{From mass,} & 0 = x+z+m, \\ \text{From time,} & 1 = -2y-z-2m. \end{array}$$

Now if we suppose $y = x$, as may very well be done, seeing that the flow is of a purely viscous nature, we are left with but one unknown, on account of peculiarities in the coefficients.

$$z = 1, \qquad m = -(x+1), \qquad n = 2x.$$

Thus

$$\mathrm{V}^{-1} = \Sigma\mathrm{A}\,(\mu\delta^x g^x \mathrm{S}^{-(x+1)}\mathrm{D}^{2x}).$$

Since the velocity increases with difference in density of the liquids, we give x the successive values -1, -2, -3, &c., and obtain the velocity expressed in a series.

As there are two coefficients of viscosity to be taken into account, the series should properly be of the form

$$\mathrm{V}^{-1} = (a\mu + b\mu')\left\{\frac{\mathrm{A}_1}{\delta g\mathrm{D}^2} + \frac{\mathrm{A}_2\mathrm{S}}{\delta^2 g^2\mathrm{D}} + \frac{\mathrm{A}_3\mathrm{S}^2}{\delta^3 g^3\mathrm{D}^6} + \&\text{c.}\right\}.$$

* The form of the series represents the reciprocal, instead of the velocity itself, because it so happened the constants were originally so calculated, and a change would involve the arithmetical labour over again.

† The dimensions of δ are $\delta = \mathrm{M/L^3}$, of $g = \mathrm{L/T^2}$, of $\mu = \mathrm{M/LT}$, and of $\mathrm{S} = \mathrm{M/T^2}$.

But in most of the experiments made the viscosity of the liquid of which the column was composed was so much greater than that of the bubble that the following form proved sufficiently accurate :—

$$V^{-1} = \frac{A_1\mu}{\delta g D^2} + \frac{A_2\mu S}{\delta^2 g^2 D^4} + \frac{A_3\mu S^2}{\delta^3 g^3 D^6} + \&c.$$

The value of these constant coefficients could be experimentally found by a series of determinations of velocity through different sized tubes, the same two substances being used throughout.

Taking only three terms of the series, I have done this for the case of air bubbles ascending through columns of glycerine of different diameters, and I find that the constants thus determined are practically *the same* as those required by my experiments with other substances. Thus three terms would appear to be ample.

In the following table is exhibited the time taken by a bubble of air to ascend 1 cm. (the reciprocal of the velocity) through a column of glycerine, the diameter of which is given in the top row.

Table I.—Air—Glycerine.

Diameter of tube..	0·609	0·665	0·775	0·895	1·03	1·28	1·46	1·68
Time observed....	40·5	15·2	7·43	3·09	1·73	0·784	0·522	0·325
Time calculated...	32·3	18·9	7·77	2·67	1·72	0·723	0·493	0·336

The third row was calculated by using the values of the constants given below, which were themselves deduced from the second row by the method of least squares.

$$A_1 = 1{\cdot}308 \,.\, g/\mu;$$
$$A_2 = 0{\cdot}02322 \,.\, g^2/\mu;$$
$$A_3 = 0{\cdot}0009108 \,.\, g^3/\mu.$$

The difference in density was 1·25, and the surface tension 63 dynes per centimetre. Temperature was that of the air, and ranged between 10° and 14°. For some sized tubes the agreement between the observed and the number calculated is not good, but this is probably due to variations in temperature. The viscosity of glycerine varies rapidly with temperature. The importance of constant temperature was not appreciated until most of the experiments made had been completed.

Having once determined the constant coefficients, it becomes possible to calculate the velocity of a bubble of any substance through a tube of glycerine of given diameter. The only things now requisite

for doing this are the difference in density of the bubble and its surface tension. An examination of the following tables will, I think, justify this assumption.

In Table II are exhibited the values of the "velocity reciprocal" calculated for chloroform in this way. That is to say, the ascertained values of the surface tension between chloroform and glycerine ($S = 12{\cdot}1$) and of the difference of their densities ($\delta = 0{\cdot}253$) were simply introduced into the expression we above obtained for the velocity. In the third row for comparison is given the time taken per centimetre found by actual experiment.

Table II.—Chloroform—Glycerine.

Diameter of tube.. ..	0·665	0·775	0·895	1·03	1·28	1·68
Time calculated.....	82·6	33·5	15·0	7·86	3·32	1·61
Time observed......	81·2	33·7	14·4	8·17	3·40	1·86

$\delta = 0{\cdot}253.$ $S = 12{\cdot}1.$

In the following two tables the same is given for creasote and mercury.

Table III.—Creasote—Glycerine.

Diameter of tube....	0·31	0·41	0·50	0·609	0·665	0·775	0·895	1·03	1·28
Time calculated.....	480	98·6	37·4	18·9	14·2	9·80	7·26	5·55	3·67
Time observed......	480	86·8	36·9	20·4	17·1	8·76	7·03	5·44	3·91

$\delta = -0{\cdot}199.$ $S = 2{\cdot}05.$

Table IV.—Mercury—Glycerine.

Diameter of tube........	0·41	0·50	0·665	1·03
Time calculated	12·6	3·15	0·495	0·097
Time observed..........	10·7	4·37	0·537	0·126

$\delta = 12{\cdot}34.$ $S = 370.$

For facility of comparison, the following table is selected from the foregoing, with the addition of one other, giving particulars in the case of two sized tubes of diameter 0·665 and 1·03 respectively.

TABLE V.

	δ.	S.	D = 0·665.		D = 1·03.	
			Time observed.	Calculated.	Time observed.	Calculated.
Mercury	12·34	370	0·537	0·495	0·126	0·097
Air..................	−1·25	63	15·2	18·9	1·73	1·72
Oil of lemons	−0·377	6·8	12·0	12·7	..	..
Chloroform	0·253	12·1	81·2	82·6	8·17	7·86
Creasote	−0·199	2·05	17·1	14·2	5·44	5·55

The agreement with theory is remarkable, considering the great range introduced into the experiments. The density difference and the surface tension vary between themselves nearly 100 times.

In the accompanying diagram are plotted the various forms the equation takes on putting in the values of the physical constants proper to each body. The abscissæ represent the diameter of the tube in millimetres, and the ordinates the time calculated to be taken by the bubble in travelling 1 cm. The times found by experiment given in the above tables are marked so far as the limits of the paper would allow for the purpose of comparison.

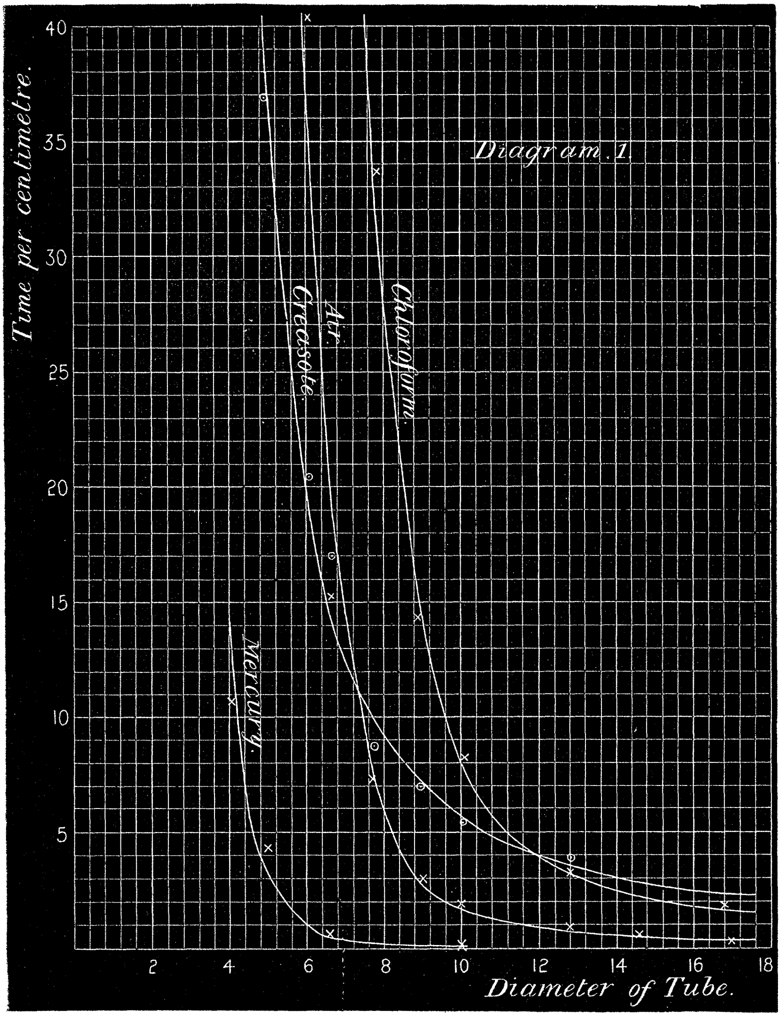
The case of creasote is instructive. With wide tubes the bubble moves comparatively slowly. For when the diameter is large, the first term in the series is most important, so that the density is then the controlling factor. But when D is small, the third term is most important, so that if the surface tension is very small, as is so in the case of creasote, the bubble will move comparatively quickly.

In such case the bubble assumes a long, almost needle-like appearance, giving ample room round it for the flow past of the liquid of the column. In the case of chloróform, for instance, this long form is not assumed because of the higher value of the surface tension.

The following table exhibits, in the case of several substances, the length of the bubble in motion relatively to its length at rest, when, of course, it occupies the whole width of the tube. The diameter of tube = 0·665.

Table VI.

Mercury	1·8
Air	1·2
Oil of lemons......	1·8
Chloroform	1·4
Creasote..........	2·0



Besides the foregoing, experiments were also made in which the viscosity of the liquid through which the bubble passed was changed in order to experimentally verify that the *velocity varied inversely as the viscosity* in agreement with the theory.

For this purpose the viscosity of glycerine was gradually reduced by the addition of water, and the "time of descent" of a chloroform bubble through a tube of the solution observed. This could be then compared with that calculated from the theory by introducing the value of the viscosity ascertained in each case.

In the following table are shown the observed and calculated "times" in the case of two sized tubes. The first column gives the approximate percentage composition of each solution, beginning with glycerine and ending with pure water. The fourth column consists of the ascertained viscosity in terms of that of water. Columns 5 and 6 are the calculated and observed "times of descent" respectively for a tube of diameter 0·655. Columns 7 and 8 are the like for a tube of diameter 0·895. Temperature between 10—12°.

Table VII.—Chloroform—Solutions of Glycerine and Water.

Per cent.	δ.	S.	μ.	D = 0·665.		D = 0·895.	
				Calc.	Observ.	Calc.	Observ.
100	0·253	12	833·0	82·6	81·2	15·0	14·4
90	0·277	14	199·0	20·5	19·1	3·73	3·35
80	0·294	*16**	64·0	7·41	6·97	1·28	1·17
70	0·320	*18*	30·0	3·32	3·31	0·58	0·52
60	0·343	20	13·0	1·45	1·53	0·25	0·4
40	0·394	*23*	5·7	0·55	0·87		
0	0·404	28	1·0	0·08	0·62		

It will be seen from this table that within certain limits we are justified in assuming the velocity to be inversely as the viscosity, but there is a complete failure in the applicability of the law in cases where the viscosity is small. The retardation observed in such cases is doubtless due to the formation of eddying motions in the fluid, and, indeed, if any small motes which happen to be present be watched, considerable commotions are always to be seen as the bubble passes whenever there is this retardation.

The retardation is even more marked in the case of air bubbles, for as will be seen from the following table, not only are the observed values greater than the calculated values, but they are actually greater than values observed with *greater* viscosities, *i.e.*, than those given higher up the list. Diameter of tube = 0·665 c.c.

This retardation is better shown in Diagram 2, which is plotted from a number of careful observations. The maximum velocity with this sized tube corresponds to about 33 per cent. of glycerine, the velocity of which is about three times that of water at the temperature of the experiments. The abscissæ represent the percentage composition of the solutions, beginning on the left with glycerine. The ordinates are the corresponding velocities.

* The values of surface tension printed in italics were interpolated.

TABLE VIII.—Air—Solutions of Glycerine and Water.

Per cent.	δ.	S.	μ.	Time calculated.	Time observed.
100	1·251	63	833	18·9	15·2
90	1·227	64	197	4·94	4·48
80	1·210	65	64	1·70	1·67
70	1·184	66	30	0·88	0·88
60	1·161	67	13	0·41	0·58
40	1·110	69	5·7	0·22	0·48
35	1·101	70	3·6	0·15	0·45
0	1	74	1	0·06	0·53

It is of interest, as pointed out by Lord Kelvin,* to consider the increase in velocity accompanying increase in viscosity here exhibited in the light of Professor Osborne Reynolds' 'Critical Velocity.' As will be remembered, the 'Critical Velocity,' or the velocity of flow of a fluid just unaccompanied by eddying motions, rises in value with rise in the value of the viscosity. On the diagram, an hypothetical line to indicate the "critical velocity" for the bubble has been drawn. (That is to say, the velocity of the bubble which is accompanied by critical velocity of flow of the fluid past it.) All calculated values of the velocity below this line will stand. Above this line the observed velocity will lie between it and the line given by the calculation.

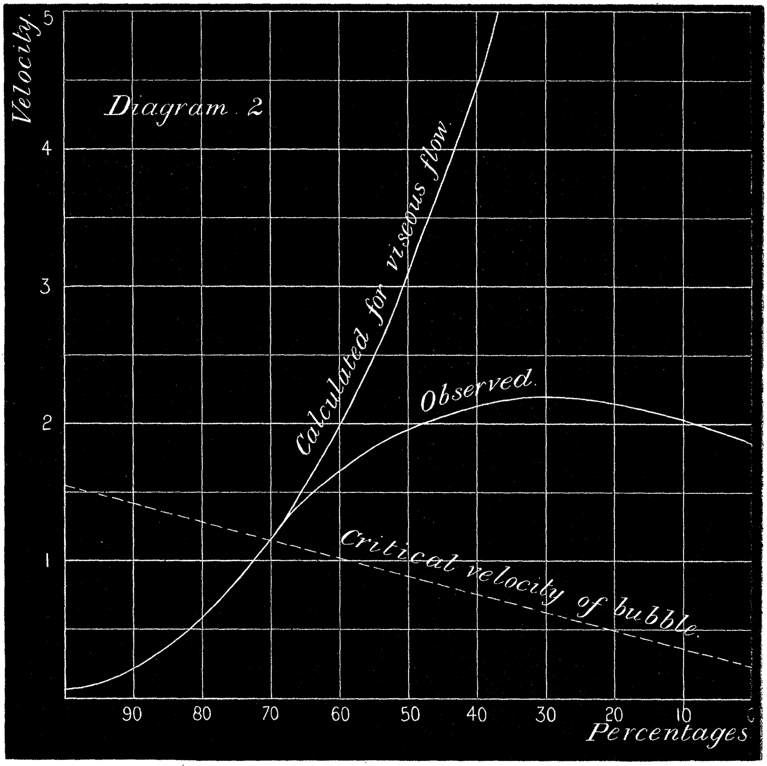
The form of the curve was found to be the same with solutions in water of sugar, of calcium chloride, or of sodium hydrate.

The actual turning down of the curve of velocities at small values of the viscosity, as occurs in these cases, would apparently be a matter of the rate of departure from each other of the critical velocity curve and that calculated from the theory of viscous flow, which latter, it must be remembered, depends on the surface tension and density as well as on the viscosity. These quantities probably have an important bearing in this particular question. By employing, say, gum tragacanth, one can increase the viscosity of water without sensible change in density.† Thus, in this respect the driving power is not simultaneously increased as is the case when increase in viscosity is produced by addition of glycerine.

With gum tragacanth the velocity never actually increases on increase of viscosity as in the case of glycerine. But through a considerable range in viscosity on the right-hand side of the diagram

* In the course of a discussion on a note read before the British Association in Edinburgh, 1892.

† The surface tension of these solutions changes in about the same relative ratio to the viscosity as solutions of glycerine in water do.



the velocity curve runs almost horizontal, showing clearly that the phenomenon of turbulency is present.

Similar curves to that for gum tragacanth were found on employing starch or gelatine to increase the viscosity of water.

The case of soap is remarkable. A very slight addition of soap to water produces quite a sensible increase in the velocity of the bubble. The first row in Table IX gives the percentage present of a certain soap solution ($\delta = 1{\cdot}026$) in mixture with water. Beneath is the velocity of the air bubble in each case. Diameter of tube was 0·661.

Table IX.—Velocity through Soap Solutions.

Per cent...	0	0·078	0·31	1·25	2·5	5·0	10·0	20·0	40·0	60·0	70·0	80·0
Velocity ..	1·86	1·90	2·39	3·68	3·76	4·13	4·18	4·25	4·46	3·52	1·26	0·74

It so happens that at first on addition of soap a diminution of viscosity does occur; nevertheless the chief cause in the increase in

velocity is probably due to the great diminution of the surface tension, which for the 5 per cent. solution was less than half that of water.

In connexion with this, as it probably depends on the formation of soap, must be mentioned a rather pretty experiment, which is easily made. A bubble of sweet oil if allowed to ascend a tube (say, of diameter = 1) of ordinary pure water passes up in the ordinary way. But, if the tube contains a weak solution of caustic soda, as the bubble ascends, the motion of the solution over the surface of the oil raises a series of circular waves round the cylindrical bubble, or, rather, a series of surface tension ripples. The waves almost invariably join at once to form one continuous spiral wave round the bubble, which then lends a surprisingly life-like appearance to the bubble as it wriggles its way upwards through the tube.*

The system of circular waves is evidently unstable, since the formation of the spiral means the opening of a continuous channel for the flow past of the solution.

The sign of the spiral (right- or left-handed) depends on initial circumstances, and can, when the tube is held in the hand, be conditioned at will by a judicious turn of the wrist.

The solution *must* be very weak, best about 1 part of strong caustic soda in 50,000 parts water; much stronger than this has too great a tendency to emulsify the oil, doubtless itself a phenomenon in part arising from diminution in surface tension on the more exposed parts of the oil.

When the liquid of which the column is composed is very viscous, the system of waves is prevented from forming, as is the case with a creasote bubble passing through glycerine, despite the fact of the surface tension being so very small.

At first sight it might appear that the ascent of bubbles through tubes of different liquids would prove a convenient method for comparing their viscosities. As has been seen, it is necessary, among other things, to know the surface tension in each case. This renders the comparison of viscosity in this way really a more troublesome process than by ordinary methods, since the determination of surface tension, especially if the liquids be viscous, is often accompanied by considerable uncertainty, owing to a persistent tendency to stick to glass frequently exhibited. This is particularly so in finding the surface tension between two liquids.

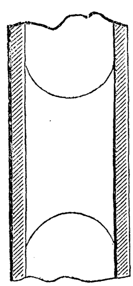
The phenomenon, however, would appear to be suited for the comparison of the surface tension between liquids. If the same liquid constitute the column in each case, it need not be necessary to know its viscosity, only the density being requisite.

* On inclining the tube, especially with smaller sized tubes, the bubble is seen to have quite a caterpillar mode of progression.

In this way the surface tension between two liquids, even though they dissolve each other in all proportions, may be measured. Thus, a bubble of water can be got to ascend a tube of glycerine, preserving for a considerable distance its perfectly distinct shape. In this fashion I have found 6·5 dynes per cent. for the initial value of the surface tension between water and glycerine. The velocity of the bubble rapidly increases as it proceeds, owing to the surface tension diminishing according as glycerine dissolves in the water. The continuous replacement of one of the liquids in the method gives it an advantage for such a purpose as this over any statical method. The quantity here discussed would in all likelihood prove to be related to molecular rates of diffusion, and perhaps should for this reason merit consideration.

An important point, which up to this has not been touched upon, is the necessity for the liquid of which the bubble is composed not to adhere so tenaciously to the walls of the tube as to cause the bubble to retain the shape shown in fig. 2 in section. This cannot

FIG. 2.



happen when the resolved component parallel to the axis of the tube of the tension of the surface between the liquids *plus* the tension at the surface of separation of the bubble and tube exceeds that between the second liquid and tube. Even though this be not the case, the bubble will often assume the shape with convex ends suitable for travelling through the tube.

With every substance, as the diameter of the column is reduced, a stage is reached when there is a tendency to stick even though the end be convex.

This limit is found to depend largely on the surface tension; with high values of surface tension it is soon reached. For instance, a bubble of mercury sticks hopelessly in a tube of glycerine of diameter 0·31, while creasote travels freely, although the pressure driving it is 62 times less. Now the surface tension in the case of mercury is 180 times that of the creasote used in these experiments.

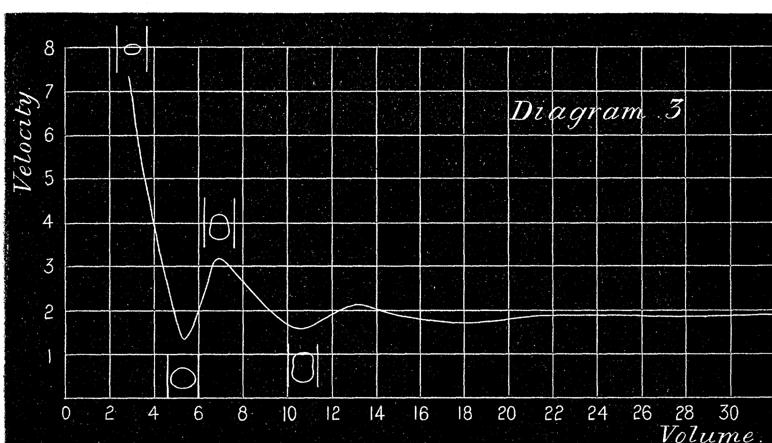
No quantitative determinations have been made on the remaining

experimentally unverified factor in our expression for the velocity of the bubble. That is to say to verify the terms under which the acceleration (g) appears in the expression. This, were it desirable, might of course be investigated by an arrangement utilising centrifugal force, but without elaborate arrangements it is not difficult to observe the great influence increase in pressure produced by swinging a tube round and round by hand has. To take an example, a bubble of creasote in a tube of glycerine of diameter = 0·31 takes over 13 hours to travel 1 m. By swinging it round and round in the hand, one can get it through in less than as many minutes.

In connection with this the comparative ease with which a stray liquid bubble in the tube of a thermometer can be brought home by swinging in the hand will suggest itself.

Part II.

When the length of the bubble is less than a certain thing, the velocity is found to be very different for different sized bubbles. The result of some experiments made with different sized bubbles of air passed up through a tube of water of diameter 0·665 is exhibited in Diagram III.



The ordinates represent the velocity and the abscissæ the volume of the bubble measured in terms of the length of tube it occupies when at rest. The length of the bubble in motion is always of course greater than this.

It will be seen that the velocity corresponding to a volume of about 7 is nearly double that at about 5·5. The phenomenon is, doubtless,

due to the form taken by the bubble. The figures on the diagram roughly represent the appearance of the bubble corresponding to the different points where they are placed. The small bubble on the left passes up rapidly. As larger volumes of air are taken to form the bubble the velocity falls to a minimum, at which point the bubble is almost spherical in shape. With further additions of air, the spherical form gives way to one pointed at top. Again more air swells out the top and gives the bubble a somewhat dumb-bell appearance. For this and the spherical form the velocity is a minimum. In these instances the ratio of the resistance of the annular channel to the flow-past of the liquid compared to the driving pressure is a maximum. The intermediate shape being pointed at top gains an increased pressure head without a corresponding increase in resistance. The various forms assumed by the bubbles remind one of the well-known initial shapes taken on the formation of liquid drops.

Similar curves were obtained with other sized tubes, the phenomenon being rather more marked in the case of the smaller sizes. Other liquids were found to behave in a like way when used either as the substance of the column or for constituting the bubble itself.

The curve in the diagram exhibits the general behaviour of the bubbles, but occasionally an anomalous determination of velocity will be obtained; this is accompanied by the bubble having also an anomalous form. That is to say, a volume of air a little in excess of that corresponding to the spherical stage, instead of assuming a pointed form at top, will retain a spherical form, and as a consequence will travel very slowly.

V. "On the Metallurgy of Lead." By J. B. HANNAY, F.R.S.E., F.I.C. Communicated by Sir G. G. STOKES, F.R.S. Received April 15, 1893.

[Publication deferred.]

Presents, June 1, 1893.

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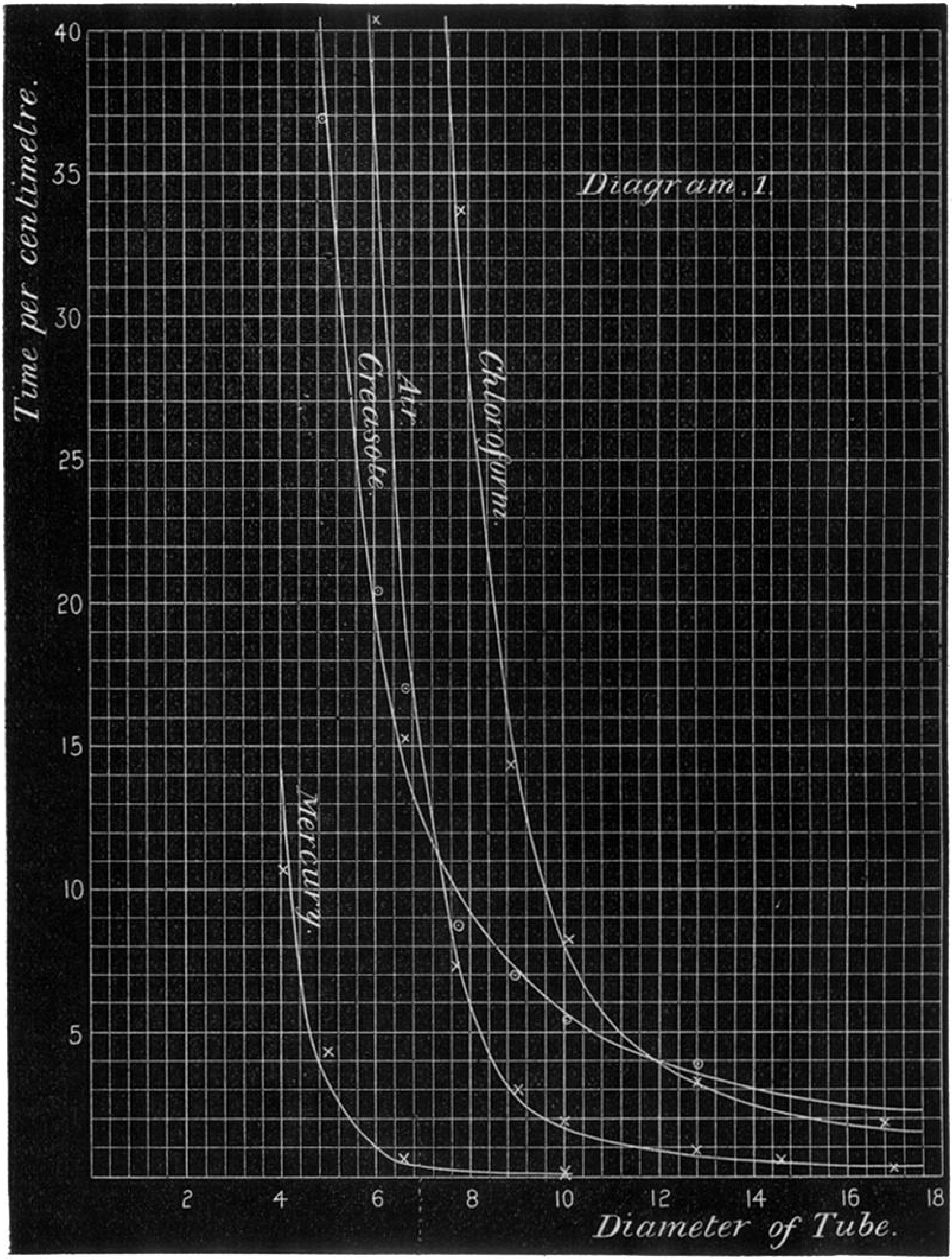

Diagram. 1.
Time per centimetre.
Diameter of Tube.
Chloroform.
Air.
Creasote.
Mercury.
40
35
30
25
20
15
10
5
2
4
6
8
10
12
14
16
18

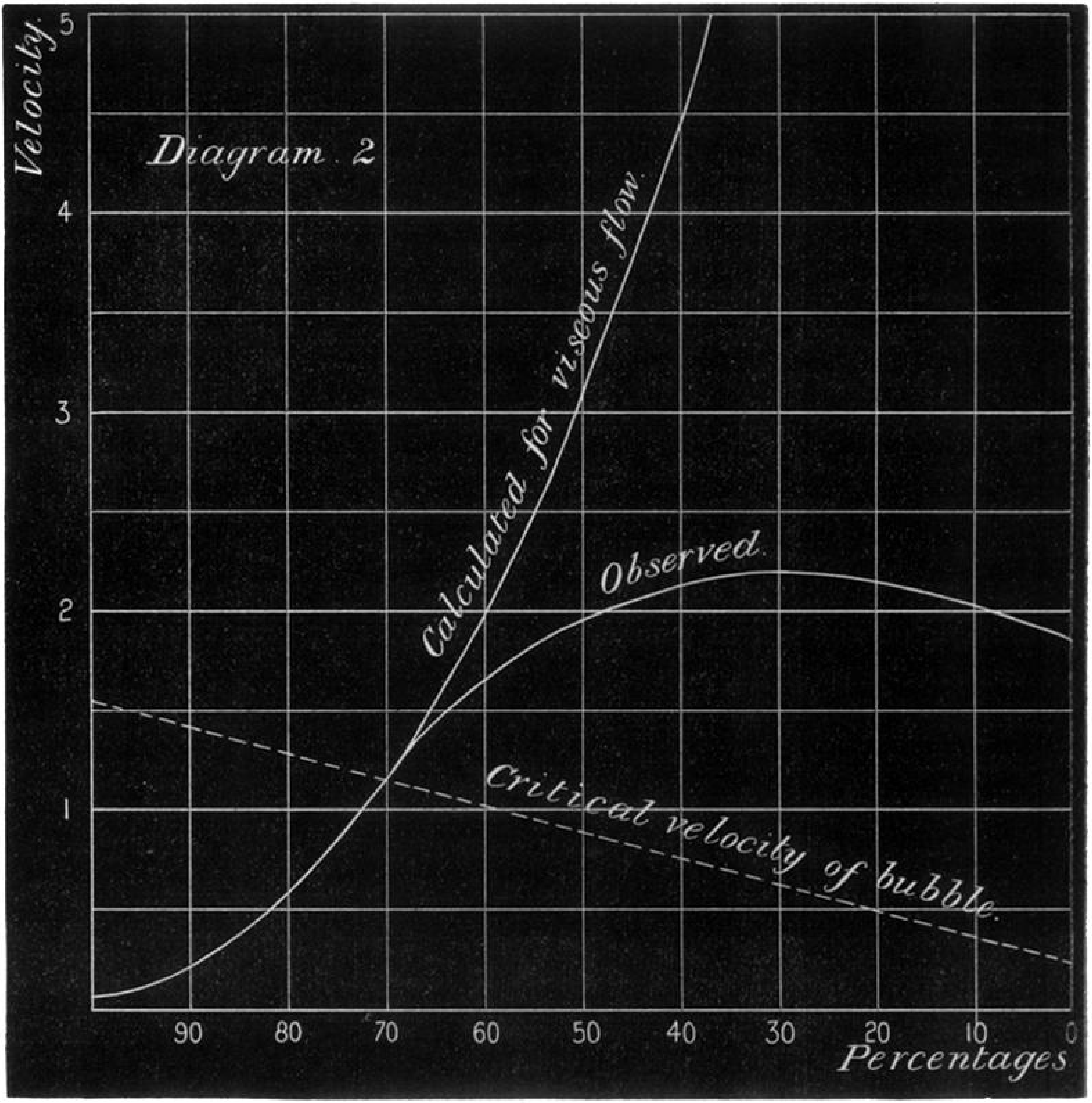
Diagram 2
Velocity.
5
4
3
2
1
Calculated for viscous flow.
Observed.
Critical velocity of bubble.
90
80
70
60
50
40
30
20
10
0
Percentages

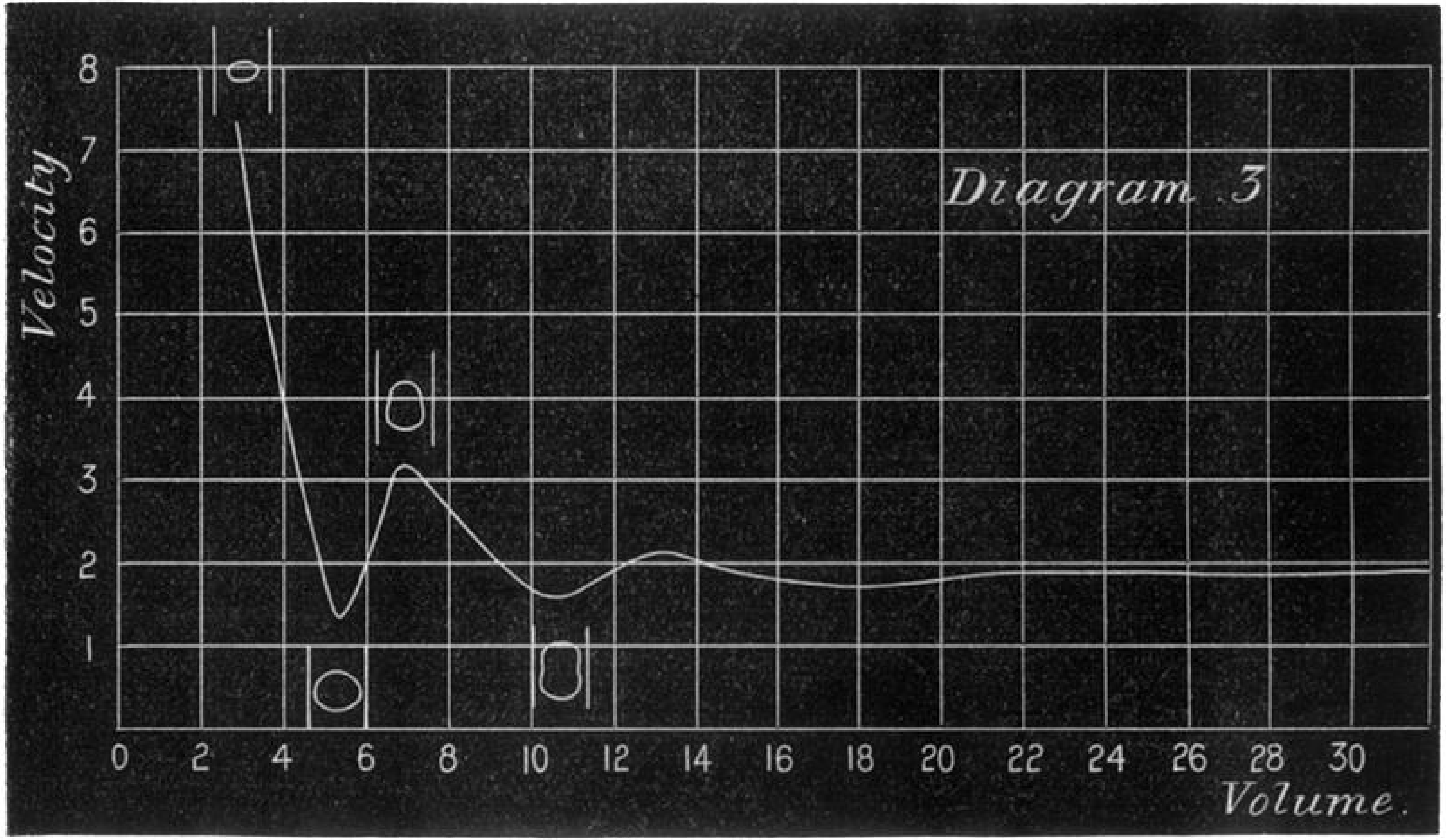
Diagram 3
Velocity
8
7
6
5
4
3
2
1
0
2
4
6
8
10
12
14
16
18
20
22
24
26
28
30
Volume.